

Codes and supersymmetry in one dimension

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Abstract

Adinkras are diagrams that describe many useful supermultiplets in $D = 1$ dimensions. We show that the topology of the Adinkra is uniquely determined by a doubly even code. Conversely, every doubly even code produces a possible topology of an Adinkra. A computation of doubly even codes results in an enumeration of these Adinkra topologies up to $N = 28$, and for minimal supermultiplets, up to $N = 32$.

1 Introduction, review and synopsis

Although many supersymmetric theories have been known since the 1970s, there is still no overarching classification of supermultiplets, even in one dimension (time). In fact, supersymmetry in one dimension has been the subject of several investigations, for instance the development of the $\mathcal{GR}(d, N)$ algebras [1–3], the development of Adinkras [4], and to other efforts [5–14].

An Adinkra is a directed graph with various colorings and other markings on vertices and edges, which in a pictorial way encode all details of the supersymmetry transformations on the component fields within a supermultiplet in one dimension [4]. The main purpose of the present work is to determine the kinds of graphs (i.e., the *topology*) that can be used for Adinkras.

The class of supermultiplets described by Adinkras is wide enough to contain many noteworthy superfields. [4, 15] This paper will clarify the conditions for a supermultiplet in one dimension to be described using an Adinkra. Beyond this, by showing that this class is large, we thereby show that the class of supermultiplets is also large, and thus establish that the classification problem of supersymmetry in one dimension is much more intricate than it might appear at first. We also thereby add many new supermultiplets to the literature that were previously unknown, and it is possible that some of these newly discovered supermultiplets may be useful or interesting in their own rights.

Finally, we hope that classifying Adinkras may help one better understand the conditions under which a superfield has an off-shell description. Subject to a particular set of assumptions about dynamics, Siegel and Roček [16] had previously shown that not all supermultiplets have off-shell descriptions. On the other hand, superfields described by Adinkras are off-shell supermultiplets. Therefore, understanding the range of what Adinkras can describe may shed some light on the question of which supermultiplets admit an off-shell description.

1.1 Adinkras

In one dimension, with N supersymmetry generators Q_1, \dots, Q_N , the supersymmetry algebra is

$$\{Q_I, Q_J\} = 2i\delta_{IJ}\partial_\tau, \quad (1.1)$$

where ∂_τ is the derivative in the time direction.

An Adinkra is a finite directed graph, with every vertex colored either white or black, and with every edge colored one of N colors (each color corresponds to one of the supersymmetries Q_I), and each edge drawn with either a solid or a dashed line. The vertices correspond to the component fields (black for fermions, white for bosons) and the edges correspond to the action of each of the Q_I , in a way that is reminiscent of the Cayley diagram of a finitely generated group, or even more analogously, the Schreier diagram of the set of cosets of a subgroup. Details of Adinkras and how they correspond to supermultiplets can be found in [4, 17]. The classification of Adinkras naturally falls into four steps:

- (1) Determine which topologies are possible (the topology of an Adinkra is the underlying graph of vertices and edges without colorings, as, for instance, in [17]).
- (2) Determine the ways in which vertices and edges may be colored. The topology of the Adinkra, together with the colorings of vertices and edges, will be called the *chromotopology* of the Adinkra. It is chromotopologies that are classified in this paper.
- (3) Determine the ways in which edges may be chosen as dashed or solid. This is closely related to the well-known theory of Clifford algebras, and will be studied in a future effort.
- (4) Determine the ways in which arrows may be directed along each edge. This issue is addressed in [17], and shown to be equivalent to the question of “hanging” the graph on a few sinks. Alternately, we can start with an Adinkra where all arrows go from bosons to fermions, then perform a sequence of vertex raises to arrive at other choices of arrow directions.

As it happens, it is convenient to do 1 and 2 together; that is, to classify chromotopologies. Herein, we show that the classification of Adinkra chromotopologies is equivalent to another interesting question from coding theory: the classification of doubly even codes. Much work has already been done in this area [18–20], and the work described in this paper goes even further in developing this classification; see Appendix B.

We emphasize that we focus here on the representation theory, not the dynamics. This is natural, as we need to first know the full palette of supersymmetric representations before discussing the properties of the dynamics in theories built upon such representations. For instance, presupposing a standard, uncoupled Lagrangian for the supermultiplets that we intend to classify would necessarily limit the possibilities; there do exist supermultiplets which can only have interactive Lagrangians [21, 22]. Herein, we defer the task of finding Lagrangians involving the supermultiplets considered in this paper. In [23, 24], we have in fact started on such studies, and, using Adinkras, have constructed supersymmetric Lagrangians for some of the supermultiplets that are also discussed herein.

In units where $\hbar = 1 = c$, all physical quantities may have at most units of mass, the exponent of which is called the *engineering dimension* and is an essential element of physics analysis in general. The engineering dimension of a field $\phi(\tau)$ will be written $[\phi]$; for more details, see Refs. [17, 25].

1.2 Main result

Our main result about the chromotopology types of Adinkras and the corresponding supermultiplets, up to direct sums, may be summarized as follows:

We define the function:

$$\varkappa(N) := \begin{cases} 0 & \text{for } N < 4, \\ 1 & \text{for } N = 4, 5, \\ 2 & \text{for } N = 6, \\ 3 & \text{for } N = 7, \\ 4 + \varkappa(N-8) & \text{for } N \geq 8, \text{ recursively.} \end{cases} \quad (1.2)$$

- (1) Every Adinkra can be separated into its connected components. (The supermultiplet corresponding to such an Adinkra breaks up into a direct sum of other supermultiplets, each of which corresponds to one of the connected components of the Adinkra).
- (2) There is a one-to-one correspondence between possible chromotopologies of connected Adinkras and doubly even codes of length N .
 - (a) Each connected chromotopology has, associated to it, a doubly even code of length N and dimension $k \leq \varkappa(N)$ that records which paths connect a vertex to itself.
 - (b) The chromotopology is then the quotient of the colored N -dimensional cube by this code. (The colored N -cube is the set of vertices and edges of the N -dimensional cube $[0, 1]^N$, with colors on the

- edges determined by which axis it is parallel to, and colors of vertices according to the number, modulo 2, of coordinates that are 1).
- (c) This quotient can be viewed as an iterated k -fold \mathbb{Z}_2 -quotient.
 - (d) These chromotopologies really do come from supermultiplets in $D = 1$ dimension, and if the arrows are chosen properly (one-hooked) we can arrange it so that it is easy to see that different codes give rise to different supermultiplets.
 - (e) Permuting the columns of a code corresponds to permuting the colors of the chromotopology, which in turn describes R -symmetries of the supermultiplet.
 - (f) There are an enormous multitude of distinct doubly-even codes for $N \leq 32$, even when counting permutation equivalent codes as the same code. Thus, there is an enormous multitude of Adinkra chromotopologies.

This paper is organized as follows: Section 2 is a brief introduction to codes, and Section 3 provides a review of Adinkras and their relationship with supermultiplets. The first major result, in Section 4, is that each Adinkra chromotopology gives rise to a doubly even code. It will be convenient to provide a few classes of examples of doubly even codes for our discussions, and to give a sense for how many doubly even codes there are, so this is done in Section 5. We then turn to the second major result: that every doubly even code actually arises as the code for an Adinkra chromotopology for a supermultiplet. This is done in Sections 6 and 7. Sections 8 and 9 discuss some consequences and directions for further research.

2 Codes

We begin with a brief introduction to the theory of codes. For a more thorough introduction to the subject, see [18–20].

We think of $\{0, 1\}$ as a group with the operation \boxplus , which is addition modulo 2, i.e., the group \mathbb{Z}_2 . For the purposes of this paper, a *code* of length N means a subgroup of $\{0, 1\}^N$.¹ Although the standard notation for an element of a cartesian product is (x_1, x_2, \dots, x_N) , in practice we frequently abandon the parentheses and the commas, so that the element $(0, 1, 1, 0, 1)$ may be written more succinctly as the codeword 01101. The components of such an N -tuple are called bits, and the N -tuple is called a word. This word is called a *codeword* if it is in the code.

¹In the coding literature, there are sometimes other more general definitions of codes. What we have described is a linear binary block code of length N .

Now, \mathbb{Z}_2 is not only a group; it is also a field, so $\{0, 1\}^N$ can be viewed as a vector space over $\{0, 1\}$. All the concepts of linear algebra then apply, but with \mathbb{R} replaced with \mathbb{Z}_2 . Elements of $\{0, 1\}^N$ may be thought of as vectors, with vector addition the operation \boxplus of bitwise addition modulo 2. Codes are then linear subspaces of $\{0, 1\}^N$. Every code has a basis, called a *generating set*, g_1, \dots, g_k , so that every codeword can be written uniquely as a sum

$$\sum_{i=1}^k x_i g_i, \quad (2.1)$$

where the coefficients x_1, \dots, x_k are each either 0 or 1. The number k is the same for every generating set for a given code, and is called the *dimension* of the code. It is common to say we have an $[N, k]$ linear code when N is the length of the codewords and k is the dimension. It is traditional to denote a generating set as an $k \times N$ matrix, where each row is an element of the generating set.

If $v \in \{0, 1\}^N$, we define the *weight* of v , written $\text{wt}(v)$, to be the number of 1s in v . For instance, the weight of 01101 is $\text{wt}(01101) = 3$.

A code is called *even* if every codeword in the code has even weight. It is called *doubly even* if every codeword in the code has weight divisible by 4. Examples of doubly even codes are given in Section 5.1 below.

If v and w are in $\{0, 1\}^N$, then $v \& w$ is defined to be the “bitwise and” of v and w : the i th bit of $v \& w$ is 1 if and only if the i th bit of v and the i th bit of w are both 1. A basic fact in $\{0, 1\}^N$ is

$$\text{wt}(v \boxplus w) = \text{wt}(v) + \text{wt}(w) - 2 \text{wt}(v \& w). \quad (2.2)$$

There is a standard inner product. If we write v and w in $\{0, 1\}^N$ as (v_1, \dots, v_N) and (w_1, \dots, w_N) , then

$$\langle v, w \rangle \equiv \sum_{i=1}^N v_i w_i \pmod{2}. \quad (2.3)$$

We call v and w orthogonal if $\langle v, w \rangle = 0$. This occurs whenever there are an even number of bit positions where both v and w are 1. Note that $\langle v, v \rangle \equiv \text{wt}(v) \pmod{2}$, and thus, when $\text{wt}(v)$ is even, v is orthogonal to itself. Also note that $\langle v, w \rangle \equiv \text{wt}(v \& w) \pmod{2}$. One important consequence for us is that if $\text{wt}(v)$ and $\text{wt}(w)$ are multiples of 4, then (2.2) implies that $\text{wt}(v \boxplus w)$ is a multiple of 4 if and only if v and w are orthogonal.

3 Supersymmetric representations and adinkras

The N -extended supersymmetry algebra without central charges in one dimension is generated by the time-derivative, ∂_τ , and the N supersymmetry generators, Q_1, \dots, Q_N , satisfying the following supersymmetry relations:

$$\{Q_I, Q_J\} = 2i \delta_{IJ} \partial_\tau, \quad [\partial_\tau, Q_I] = 0, \quad I, J = 1, \dots, N. \quad (3.1)$$

In this section, we determine some essential facts about the transformation rules of these operators on fields for which it is possible to maintain the physically motivated concept of engineering dimension. We note that since the time-derivative has engineering dimension $[\partial_\tau] = 1$, the supersymmetry relations (3.1) imply that the engineering dimension of the supersymmetry generators is $[Q_I] = \frac{1}{2}$.

3.1 Supermultiplets as representations of supersymmetry

A real supermultiplet \mathcal{M} is a real, finite-dimensional, linear representation of the algebra (3.1), in the following sense: It is spanned by a basis of real bosonic and fermionic *component fields*, $\phi_1(\tau), \dots, \phi_m(\tau)$ and $\psi_1(\tau), \dots, \psi_n(\tau)$, respectively; each component field is a function of time, τ . The supersymmetry transformations, generated by the Hermitian operators Q_1, \dots, Q_N , act linearly on \mathcal{M} while satisfying equations (3.1). The supermultiplet is *off-shell* if no differential equation is imposed on it². The number of bosons as fermions is then the same, guaranteed by supersymmetry.

3.2 Building supermultiplets from Adinkras

The authors of [4, 17, 23] introduced and then studied Adinkras, diagrams that encode the transformation rules of the component fields under the action of the supersymmetry generators Q_1, \dots, Q_N .

Supermultiplets that can be described by Adinkras have a collection of bosonic and fermionic component fields and a collection of supersymmetry

²Logically, it is possible for some — but not all — component fields to become subject to a differential equation. This does not violate the literal definition of the off-shell supermultiplet. However, it does obstruct standard methods of quantization, which is our eventual purpose for keeping supermultiplets off-shell. For an example in 4-dimensional supersymmetry; see [15].

generators Q_1, \dots, Q_N , so that: (1) Given a bosonic field ϕ and a supersymmetry generator Q_I , the transformation rule for Q_I of ϕ is of the form

$$\text{either } Q_I \phi = \pm \psi, \quad (3.2)$$

$$\text{or } Q_I \phi = \pm \partial_\tau \psi, \quad (3.3)$$

for some fermionic field ψ . (2) Given instead a fermionic field η and a supersymmetry generator Q_I , the transformation rule of Q_I on η is of the form

$$\text{either } Q_I \eta = \pm i B, \quad (3.4)$$

$$\text{or } Q_I \eta = \pm i \partial_\tau B, \quad (3.5)$$

for some bosonic field B . In particular, these supersymmetry generators act linearly using first-order differential operators. Furthermore, the supersymmetry algebra requires that

$$Q_I \phi = \pm \psi \iff Q_I \psi = \pm i \partial_\tau \phi, \quad (3.6)$$

and

$$Q_I \psi = \pm \partial_\tau \phi \iff Q_I \phi = \pm i \psi, \quad (3.7)$$

and where the \pm signs are correlated to preserve equations (3.1).

More generally, suppose we label the bosons ϕ_1, \dots, ϕ_m and the fermions ψ_1, \dots, ψ_m . Choose an integer I with $1 \leq I \leq N$, and an integer A with $1 \leq A \leq m$. For each such pair of integers, we consider the transformation rules for Q_I on the boson ϕ_A , and we might expect that these will be of the form

$$Q_I \phi_A(\tau) = c \partial_\tau^\lambda \psi_B(\tau), \quad (3.8)$$

where $c = \pm 1$, $\lambda = 0$ or 1 , and B is an integer with $1 \leq B \leq m$, so that ψ_B is some fermion; each of c, λ, B will, in general, depend on I and A . Note that

$$\lambda = [\phi_A] - [\psi_B] + \frac{1}{2}, \quad (3.9)$$

